

ON CERTAIN FREE PRODUCT FACTORS VIA AN EXTENDED MATRIX MODEL

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ABSTRACT. A random matrix model for freeness is extended and used to investigate free products of free group factors with matrix algebras and with the hyperfinite II_1 -factor. The latter is shown to be isomorphic to a free group factor having one additional generator.

Introduction.

The finite von Neumann algebra $L(G)$ associated to a discrete group G was introduced by Murray and von Neumann [5]. It is the von Neumann algebra generated by the representation of G on $l^2(G)$ by left translation operators, with faithful trace given by the vector-state for δ_e . They gave the free group factors $L(\mathbf{F}_K)$ (where \mathbf{F}_K is the nonabelian free group on K generators), ($2 \leq K \leq \infty$) as examples of type II_1 factors which are not hyperfinite. Much of the structure around free group factors is not yet understood. For example, the isomorphism question of whether $L(\mathbf{F}_n) \cong L(\mathbf{F}_m)$ for $n \neq m$ remains open. In another direction, one can ask: for which groups G is $L(G)$ a free group factor?

Voiculescu [8,9,10,11,12] has created a theory of freeness in noncommutative probability spaces, which has begun to shed light around the free group factors [10,6]. In this theory there is a notion of the free product of finite von Neumann algebras with specified traces, (see also [2]), for which one has $L(G_1) * L(G_2) \cong L(G_1 * G_2)$.

Murray and von Neumann showed also in [5] that there is up to isomorphism only one hyperfinite factor of type II_1 , which we denote by R . In his fundamental paper [3], Alain Connes has shown that $L(G) \cong R$ if and only if G is an amenable i.c.c. group.

In this paper, we shall show that

- (a) $L(\mathbf{F}_K) * M_N(\mathbf{C}) \cong L(\mathbf{F}_{N^2K}) \otimes M_N(\mathbf{C}) \quad (1 \leq K \leq \infty, N \geq 2)$
- (b) $L(\mathbf{F}_K) * R \cong L(\mathbf{F}_{K+1}) \quad (1 \leq K \leq \infty).$

(Of course we have $\mathbf{F}_1 = \mathbf{Z}$.) This together with Connes' results implies, for example, that $L(\mathbf{Z} * G) \cong L(\mathbf{F}_2)$ whenever G is an amenable i.c.c. group.

In order to prove (a) and (b), we use random matrices. Wigner [14,15] showed that certain self-adjoint $n \times n$ random matrices with independent entries have distributions tending to a semicircle law as $n \rightarrow \infty$. (See [1,4,7,13] for a sample of the literature about random matrices.) In [11], Voiculescu showed that families of such random matrices having mutually independent Gaussian entries become, together with diagonal matrices of constant random variables, asymptotically free in the limit as $n \rightarrow \infty$. (He then used this matrix model in [10] to prove certain isomorphisms around free group factors.) We will extend this matrix model simultaneously in two respects. First,

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we may replace the Gaussian entries of the matrices in Voiculescu's theorem with more general random variables, including all cases which Wigner considered. Secondly, we may replace Voiculescu's constant diagonal matrices with constant block diagonal matrices, such that the block size remains constant (or grows slowly) as $n \rightarrow \infty$.

This paper has four sections. In §1, we give some preliminaries about freeness; in §2, we prove the extended matrix model; in §3 we prove (a) and find one more algebra isomorphic to these two; in §4 we prove (b).

§1. Preliminaries

We give here some aspects of Voiculescu's theory of freeness which we will use later. For more details, see [10,12] and other papers of Voiculescu.

A $*$ -noncommutative probability space is (A, ϕ) , where A is a unital $*$ -algebra over \mathbf{C} and ϕ a positive state on A sending 1 to 1. A W^* -noncommutative probability space is where A is a von Neumann algebra and ϕ is ultra-weakly continuous. (In addition, we will always have that ϕ is a faithful trace.) *Random variables* are elements $a \in A$. The *distribution* of $a \in A$ is the linear functional, $\mu_a : \mathbf{C}[X] \rightarrow \mathbf{C}$ which sends the polynomial, P , to $\phi(P(a))$. In the W^* setting, if a is normal, then μ_a is given by

$$\mu_a(P) = \int_{\sigma(a)} P(w) d\lambda(w),$$

where $d\lambda$ is equal to ϕ of spectral measure, supported on the spectrum of a . A family, $(a_i)_{i \in I}$ of random variables in A has a *joint distribution*, the linear functional $\mu : \mathbf{C}\langle X_i \mid i \in I \rangle \rightarrow \mathbf{C}$ (where $\mathbf{C}\langle X_i \mid i \in I \rangle$ is the algebra of noncommutative polynomials), which is $\mu = \phi \circ h$, where $h : \mathbf{C}\langle X_i \mid i \in I \rangle \rightarrow A$ is the unique unital homomorphism sending X_i to a_i . A sequence, μ_n , of (joint) distributions is said to converge to the *limit distribution*, μ , if $\lim_{n \rightarrow \infty} \mu_n(P) = \mu(P) \forall P \in \mathbf{C}\langle X_i \mid i \in I \rangle$.

A family, $(A_i)_{i \in I}$, of unital subalgebras of A is *free* if $\phi(a_1 \cdots a_n) = 0$ whenever $a_j \in A_{i_j}$, $i_1 \neq i_2 \neq \cdots \neq i_n$ and $\phi(a_j) = 0 \forall j$. A family of subsets of A (or of random variables of A) is said to be free if the unital subalgebras they generate are free, and $*$ -free if the unital $*$ -algebras they generate are free.

Proposition 1.1. ([12]).

- (i) If $(A_i)_{i \in I}$ is a free family of subalgebras in (A, ϕ) and $(B_j)_{j \in J}$ are the subalgebras generated by $\cup_{i \in I_j} A_i$, where $I = \coprod_{j \in J} I_j$ is a partition of I , then the family $(B_j)_{j \in J}$ is free.
- (ii) If $(A_i)_{i \in I}$ is a free family of subalgebras in (A, ϕ) and $(C_{ik})_{k \in K_i}$ is a free family of subalgebras in $(A_i, \phi|_{A_i})$, then the family of subalgebras $(C_{ik})_{(i,k) \in K}$ is free in (A, ϕ) , where $K = \coprod_{i \in I} \{i\} \times K_i$.

A sequence, $(a_i(n))_{i \in I}$, for $n \in \mathbf{N}$, of families of random variables is said to be *asymptotically free* if their distributions converge to a limit distribution, μ , and the random variables $(X_i)_{i \in I}$ in $(\mathbf{C}\langle X_i \mid i \in I \rangle, \mu)$ are free.

If (A, ϕ) and (B, ψ) are $*$ -noncommutative probability spaces, then there is a state, denoted $\phi * \psi$ on the free product $*$ -algebra, $A * B$, such that, A and B being identified with canonical subalgebras of $A * B$, we have $\phi * \psi|_A = \phi$ and $\phi * \psi|_B = \psi$, and A and B are free in $(A * B, \phi * \psi)$. If A and B are von Neumann algebras, the ultra-weak closure of the GNS-representation of $(A * B, \phi * \psi)$ is what we call the *free product* of the von Neumann algebras (with specified states).

Note that, since a von Neumann algebra, M , with faithful trace, ϕ , can be faithfully represented on $L^2(M, \phi)$, its isomorphism class is completely specified by the joint distribution of a set of generators of M (together with their adjoints). Thus, for example, (utilizing the functional calculus) we see that any such von Neumann algebra generated by a $*$ -free set of K normal elements whose traces of spectral measure have no atoms is $L(\mathbf{F}_K)$.

A random variable, a , in a $*$ -noncommutative probability space is said to be *semicircular* (of radius r) if it is self-adjoint and its distribution is given by

$$\mu_a(X^m) = \frac{2}{\pi r^2} \int_0^r t^m (r^2 - t^2)^{1/2} dt \quad \forall m \geq 0.$$

it is said to be *quarter-circular* (of radius r) if it is positive and its distribution is given by

$$\mu_a(X^m) = \frac{4}{\pi r^2} \int_0^r t^m (r^2 - t^2)^{1/2} dt \quad \forall m \geq 0$$

it is said to be *circular* (of radius r) if $\left\{ \frac{a+a^*}{\sqrt{2}}, \frac{a-a^*}{i\sqrt{2}} \right\}$ is a free pair of random variables, each semicircular of radius r . A *Haar unitary* is a unitary u in A such that $\phi(u^k) = 0 \quad \forall k \neq 0$. This is equivalent to ϕ of the spectral measure of u being Haar measure on the circle.

Theorem 1.2. ([10]). *In a $*$ -noncommutative probability space (A, ϕ) , where A is a von Neumann algebra and ϕ a faithful trace, $a \in A$ is circular if and only if its polar decomposition is $a = v|a|$, where $|a|$ is quarter-circular, v is a Haar unitary and $\{v, |a|\}$ is $*$ -free.*

§2. Random matrices

The context for our study of random matrices will be the following. Let σ be a probability measure on some measure space without atoms and let $L = \bigcap_{1 \leq p < \infty} L^p(\sigma)$ (the algebra of complex valued random variables having all moments bounded) be endowed with the state E , given by $Ef = \int f d\sigma$. The $*$ -algebra of $n \times n$ random matrices is $M_n(L) = L \otimes M_n(\mathbf{C})$, with trace $\phi_n = E \otimes \tau_n$, where τ_n is the normalized trace on $M_n(\mathbf{C})$. For notation, let $\{e(i, j; n) \mid 1 \leq i, j \leq n\}$ be a system of matrix units in $M_n(\mathbf{C})$, and write for the $n \times n$ random matrix, A , having a_{ij} for (i, j) th entry

$$A = \sum_{1 \leq i, j \leq n} a_{ij} e(i, j; n), \quad (1)$$

(i.e. for convenience we omit the tensor product symbol). Thus we have

$$\phi_n(A) = n^{-1} \sum_{1 \leq k \leq n} E(a_{kk}),$$

We call $\mathbf{C} \otimes M_n(\mathbf{C}) \subseteq L \otimes M_n(\mathbf{C})$ the constant matrices, and we work with the block-diagonal constant matrices defined as follows. If N is a positive integer dividing n , let $\Delta(N, n) \subseteq \mathbf{C} \otimes M_n(\mathbf{C})$ be n/N copies of $M_N(\mathbf{C})$ embedded down the diagonal, i.e. matrices of the form (1) such that each a_{ij} is a constant and equals zero unless there are $0 \leq b \leq (n/N) - 1$ and $1 \leq \hat{i}, \hat{j} \leq N$ such that $i = bN + \hat{i}$ and $j = bN + \hat{j}$.

For an $n \times n$ random matrix, $A = \sum_{i,j} a_{ij} e(i, j; n)$, denote by $E(A)$ the constant matrix whose (i, j) th entry is $E(a_{ij})$. Let

$$\begin{aligned} [A]_\phi &= A - \phi(A)I, \\ [A]_E &= A - E(A), \\ [A]_r &= E(A) - \phi(A)I. \end{aligned}$$

Theorem 2.1. *Let*

$$Y(s, n) = \sum_{1 \leq i, j \leq n} a(i, j; s, n) e(i, j; n) \in M_n(L)$$

be random matrices for s taking values in some set S such that

$$\left. \begin{aligned} a(i, j; s, n) &= \overline{a(j, i; s, n)} \\ E(a(i, j; s, n)) &= 0 \\ E(|a(i, j; s, n)|^2) &= \frac{1}{n} \end{aligned} \right\} \quad (1 \leq i \leq j \leq n, s \in S) \quad (2)$$

$$\sup_{1 \leq i \leq j \leq n} E(|a(i, j; s, n)|^m) = O(n^{-m/2}) \text{ as } n \rightarrow \infty \quad (m \geq 1, s \in S), \quad (3)$$

that

$$\{\{a(i, j; s, n) \mid s \in S\} \mid 1 \leq i \leq j \leq n\} \quad (4)$$

is an independent family of sets of random variables and that

$$E(a(i, j; s, n)a(j, i; s', n)) = 0 \text{ whenever } s \neq s'. \quad (5)$$

In addition, fixing N and considering n which are multiples of N , for each t taking values in some set, T , let

$$D(t, n) = \sum_{\substack{0 \leq b \leq (n/N)-1 \\ 1 \leq i, j \leq N}} d(Nb + \hat{i}, Nb + \hat{j}; t, n) e(Nb + \hat{i}, Nb + \hat{j}; n)$$

be elements of $\Delta(N, n)$, having a limit distribution as $n \rightarrow \infty$ and such that

$$\sup_{n, i, j} |d(i, j; t, n)| < \infty.$$

Assume also that for each $t_1, t_2 \in T$, there exists $t_3 \in T$ such that $D(t_1, n)D(t_2, n) = D(t_3, n) \forall n$.

Then the family of sets of random variables

$$(\{Y(s, n)\}_{s \in S}, \{D(t, n) \mid t \in T\})$$

is asymptotically free as $n \rightarrow \infty$ passing through multiples of N , and moreover each $Y(s, n)$ has for limit distribution Wigner's semicircle law (of radius 2).

*Proof.*¹ We may assume without loss of generality that for some t , $D(t, n) = I_n \forall n$, and that for each $t \in T$ and convergent sequences $\{\gamma_n\}_1^\infty$ and $\{\lambda_n\}_1^\infty$ in \mathbf{C} , there exists $t' \in T$ such that $\gamma_n D(t, n) - \lambda_n I_n = D(t', n) \forall n$. Hence to show asymptotic freeness, it suffices to show that if $s_1, \dots, s_r \in S$, $p_1, \dots, p_r \geq 1$ and $t_1, \dots, t_r \in T$ are such that for each $1 \leq \iota \leq r$, either $\phi_n(D(t_\iota, n)) = 0 \forall n$ or $D(t_\iota, n) = I_n \forall n$, and in the latter case, $s_{\iota-1} \neq s_\iota$ (by convention $s_0 = s_r$), then

$$\phi_n(D(t_1, n) [Y(s_1, n)^{p_1}]_\phi \cdots D(t_r, n) [Y(s_r, n)^{p_r}]_\phi) \quad (6)$$

approaches zero as $n \rightarrow \infty$.

The proof that (6) tends to zero is combinatorial and the heart of it is contained in Lemmas 2.2, 2.5 and 2.6. We shall see that a sharp enough counting argument is possible to show that (6) goes to zero if we replace each $[Y(s_\iota, n)^{p_\iota}]_\phi$ by $[Y(s_\iota, n)^{p_\iota}]_E$. Hence we need also show that the remainder $[Y(s_\iota, n)^{p_\iota}]_r$ goes to zero fast enough. Formally, we write

$$[Y(s_\iota, n)^{p_\iota}]_\phi = [Y(s_\iota, n)^{p_\iota}]_E + [Y(s_\iota, n)^{p_\iota}]_r, \quad (7)$$

substitute (7) into (6), expand and apply Lemmas 2.5 and 2.6.

Lemma 2.3 shows that the $Y(s, n)$ all have the same limit distribution, call it μ . To see that μ is a semicircle law, one can prove it directly (see Remark 2.4); otherwise, one can use the central limit theorem as follows. Taking S equal to the set, \mathbf{N} , of natural numbers, for $M \geq 1$, let

$$\psi(M, n) = M^{-1/2} \sum_{1 \leq s \leq M} Y(s, n).$$

Then $\psi(M, n)$ for $n \in \mathbf{N}$ is a sequence of random matrices of the same type as the $Y(s, n)$'s and thus has limit distribution, μ . But by the central limit theorem for free random variables (see [8]), the distributions of the $\psi(M, n)$'s tend to a semicircle law as $M \rightarrow \infty$. Hence μ is a semicircle law. \square

¹ Another proof of Gaussian cases is described in the appendix

Lemma 2.2. *Letting $Y(s, n)$ be as in Theorem 2.1 and fixing $s \in S$ and $m \geq 1$, write*

$$[Y(s, n)^m]_r = \sum_{1 \leq i, j \leq n} f(i, j; n) e(i, j; n).$$

Then

$$\begin{aligned} \sup_{1 \leq i < j \leq n} |f(i, j; n)| &= O(n^{-3/2}) \\ \sup_{1 \leq k \leq n} |f(k, k; n)| &= O(n^{-1/2}) \end{aligned}$$

as $n \rightarrow \infty$.

Proof. Let $1 \leq x, y \leq n$. Then the (x, y) th element of $E(Y(s, n)^m)$ is

$$(E(Y(s, n)^m))_{xy} = \sum_{\substack{i_2, \dots, i_m \\ j_k = i_{k+1}}} E(a(x, j_1; s, n) a(i_2, j_2; s, n) \cdots a(i_{m-1}, j_{m-1}; s, n) a(i_m, y; s, n)), \quad (8)$$

where the sum is taken over all $i_2, \dots, i_m \in \{1, \dots, n\}$ and j_k is assigned the value i_{k+1} for $1 \leq k \leq m-1$. For ease of notation, we will frequently denote x by i_1 and y by j_m .

In order to evaluate an expression of the form

$$E(a(i_1, j_1; s, n) a(i_2, j_2; s, n) \cdots a(i_{m-1}, j_{m-1}; s, n) a(i_m, j_m; s, n)) \quad (9)$$

for a given choice of i_1, \dots, i_m and j_1, \dots, j_m , one uses the independence condition, (4), and gathers the $a(i_k, j_k; s, n)$'s together into mutually independent groups corresponding to the different (unordered) sets $\{i_k, j_k\}$ which appear. Then one multiplies together the expectations of each of the groups. Thus given a choice of $i_1, \dots, i_m, j_1, \dots, j_m \in \{1, \dots, n\}$, we define the resulting *gathering*, \mathcal{Q} , to be the pair, $(\mathcal{R}, \mathcal{O})$, where \mathcal{R} is the equivalence relation on $\{1, \dots, m\}$ indicating when $a(i_k, j_k; s, n)$ and $a(i_{k'}, j_{k'}; s, n)$ belong to the same group, and \mathcal{O} (the “orientation map” of the gathering) indicates whether $a(i_k, j_k; s, n)$ and $a(i_{k'}, j_{k'}; s, n)$ are on the same or opposite sides of the diagonal. Specifically, we let

$$k \stackrel{\mathcal{R}}{\sim} k' \text{ if and only if either (I) } \begin{matrix} i_k = i_{k'} \\ \text{and} \\ j_k = j_{k'} \end{matrix} \text{ or (II) } \begin{matrix} i_k = j_{k'} \\ \text{and} \\ j_k = i_{k'} \end{matrix} \quad (10)$$

and define

$$\mathcal{O} : \{(k, k') \in \{1, \dots, m\}^2 \mid k \stackrel{\mathcal{R}}{\sim} k'\} \rightarrow \{\pm 1\}$$

to be

$$\mathcal{O}(k, k') = \begin{cases} +1 & \text{if (I)} \\ -1 & \text{if (II) but not (I).} \end{cases} \quad (11)$$

We say that a gathering, \mathcal{Q} , has property P if each equivalence class of \mathcal{R} has at least two elements, and note that the quantity (9) is nonzero only when its gathering has P, (because each $a(i, j; s, n)$ has zero expectation).

Thus the sum in equation (8) becomes

$$(E(Y(s, n)^m))_{xy} = \sum_{\mathcal{Q} \text{ with P}} \sum_{\substack{i_2, \dots, i_m \\ j_k = i_{k+1} \\ \text{giving } \mathcal{Q}}} E(a(x, j_1; s, n) \cdots a(i_m, y; s, n)), \quad (12)$$

where the sums are over all possible gatherings, \mathcal{Q} with property P and all choices of $i_2, \dots, i_m \in \{1, \dots, n\}$ which result in gathering \mathcal{Q} . (We set $j_k = i_{k+1}$ for $1 \leq k \leq m-1$.)

Let us say that a gathering, \mathcal{Q} has property P₃ if it has property P and its equivalence relation, \mathcal{R} , also has an equivalence class of at least three elements. It has property P₄ if it has property P but not P₂ and also its orientation

map, \mathcal{O} , takes on the value $+1$ for some pair (k, k') , where $k \neq k'$. Those that are left (i.e. those gatherings with property P but not P_3 or P_4) we say have property P_5 . The sum over \mathcal{Q} with P may be split into sums over \mathcal{Q} with P_3 , P_4 and P_5 . We shall show that the first two go to zero fast enough as $n \rightarrow \infty$, and the sum over \mathcal{Q} with P_5 approaches the value $\phi_n(Y(s, n)^m)$ when $x = y$.

Using independence and the bound on moments, (3), we have $|E(a(i_1, j_1; s, n) \cdots a(i_m, j_m; s, n))| = O(n^{-m/2})$ independent of the choice of i 's and j 's. Hence for a term corresponding to a fixed \mathcal{Q} in equation (12),

$$\sup_{1 \leq x \leq y \leq n} \left| \sum_{\substack{i_2, \dots, i_m \\ j_k = i_{k+1} \\ \text{giving } \mathcal{Q}}} E(a(x, j_1; s, n) \cdots a(i_m, y; s, n)) \right| = O(n^{-m/2}) \theta_{xy}(\mathcal{Q}), \quad (13)$$

where $\theta_{xy}(\mathcal{Q})$ is the number of choices of i_1, \dots, i_m which give \mathcal{Q} . (Recall we set $j_k = i_{k+1}$ for $1 \leq k \leq m-1$).

A bound for $\theta_{xy}(\mathcal{Q})$ may be found by doing computations using Feynmann graphs (similarly to their use in [11]). Consider the straight line graph (Figure 1). Associate the k th edge with $a(i_k, j_k; s, n)$ and the k th vertex with $i_k = j_{k-1}$ for $2 \leq k \leq m$, with $x = i_1$ for $k = 1$ and with $y = j_m$ for $k = m+1$. Then

$$\theta_{xy}(\mathcal{Q}) \leq n^{d(\mathcal{Q}) - l_{xy}}, \quad (14)$$

where $d(\mathcal{Q})$ is the number of vertices in the quotient graph, G , obtained from the straight line graph by identifying edges $[k, k+1]$ and $[k', k'+1]$ if $k \stackrel{\mathcal{R}}{\sim} k'$ (i.e. if $a(i_k, j_k; s, n)$ and $a(i_{k'}, j_{k'}; s, n)$ are gathered by \mathcal{Q} into the same group), with orientation preserved if $\mathcal{O}(k, k') = +1$ and reversed if $\mathcal{O}(k, k') = -1$. We are thus identifying vertices which are forced by \mathcal{Q} to be labeled with the same value of $\{1, \dots, n\}$.

More explicitly, $d(\mathcal{Q})$ is the number of equivalence classes in $\{1, \dots, m+1\}$ modulo the equivalence relation $\stackrel{v}{\sim}$, which is generated by the relations

$$\begin{aligned} & \left\{ \begin{array}{l} k \stackrel{v}{\sim} k' \\ k+1 \stackrel{v}{\sim} k'+1 \end{array} \mid k \stackrel{\mathcal{R}}{\sim} k' \text{ and } \mathcal{O}(k, k') = +1 \right\} \cup \\ & \cup \left\{ \begin{array}{l} k \stackrel{v}{\sim} k'+1 \\ k+1 \stackrel{v}{\sim} k' \end{array} \mid k \stackrel{\mathcal{R}}{\sim} k' \text{ and } \mathcal{O}(k, k') = -1 \right\}. \end{aligned} \quad (15)$$

The fact that any assignment of values to i_1, \dots, i_m which gives \mathcal{Q} must be constant on the equivalence classes of $\stackrel{v}{\sim}$ justifies equation (14), when we also note that we may subtract l_{xy} from $d(\mathcal{Q})$ in (14) to account for the fact that $i_1 = x$ and $j_m = y$ are already assigned. We may subtract 2 if $1 \not\stackrel{v}{\sim} m$ and 1 if $1 \stackrel{v}{\sim} m$, (which would imply that $x = y$). Thus let

$$l_{xy} = \begin{cases} 2 & \text{if } x \neq y \\ 1 & \text{if } x = y. \end{cases}$$

Suppose that \mathcal{Q} has property P_3 . Then G has at most $\frac{m-1}{2}$ edges, hence (as G is connected) at most $\frac{m+1}{2}$ vertices, so

$$\theta_{xy}(\mathcal{Q}) \leq n^{\frac{m+1}{2} - l_{xy}}.$$

Suppose \mathcal{Q} has property P_4 . Then a more intricate analysis of the identifications is called for in order to bound $d(\mathcal{Q})$. \mathcal{R} groups the edges into pairs. Referring to (15), observe that when $k \stackrel{\mathcal{R}}{\sim} k'$, the identification of the edges $[k, k+1]$ and $[k', k'+1]$ results in each of the vertices being paired with a vertex not equal to itself except if the edges lie next to each other and are identified with reverse orientation (i.e. $|k - k'| = 1$ and $\mathcal{O}(k, k') = -1$). For example, taking $k = 3$ and $k' = 4$, such an identification of neighboring edges with reverse orientation results in the graph shown in Figure 2. We call this a *click* and we also use this word as a verb. A click removes two vertices

from the line and leaves a tail of one vertex hanging below. Now, if possible, we click again, clicking any two edges of the line which are presently neighbors and are identified by \mathcal{Q} with reverse orientation. After c clicks the graph will consist of a line with $m - 2c + 1$ vertices together with tails hanging below which have a total of c additional vertices. After we've clicked as much as possible, there will still remain some edges above on the line, because of the requirement that at least one pairing be orientation preserving. And then each of the vertices remaining on the line will become identified with some vertex not equal to itself when we do the further identifications. Hence G will have at most $\frac{m-2c+1}{2} + c = \frac{m+1}{2}$ vertices and

$$\theta_{xy}(\mathcal{Q}) \leq n^{\frac{m+1}{2} - l_{xy}}.$$

So equation (12) becomes

$$(E(Y(s, n)^m))_{xy} = S_5(x, y; m, n) + O(n^{1/2})n^{-l_{xy}}, \quad (16)$$

where

$$S_5(x, y; m, n) = \sum_{\mathcal{Q} \text{ with } P_5} \sum_{\substack{i_2, \dots, i_m \\ j_k = i_{k+1} \\ \text{giving } \mathcal{Q}}} E(a(x, j_1; s, n) \cdots a(i_m, y; s, n)) = n^{-m/2} \sum_{\mathcal{Q} \text{ with } P_5} \theta_{xy}(\mathcal{Q}) \quad (17)$$

and where $O(n^{1/2})$ is uniform with respect to x and y . (The last equality of (17) results from equation (2), independence and the fact that \mathcal{Q} with P_5 groups the $a(i, j; s, n)$'s into complex conjugate pairs.) The value of $S_5(x, y; m, n)$ thus depends only on whether $x = y$ or $x \neq y$ and not on the particular values of x and y . It's easy to see that $\theta_{xy}(\mathcal{Q}) = 0$ when $x \neq y$ (and \mathcal{Q} has P_5). (Indeed, an induction argument shows, for example, that if we let M be the number of elements (i, j) in the list $(x, j_1), (i_2, j_2), \dots, (i_{m-1}, j_{m-1}), (i_m, y)$ for which exactly one of i and j is equal to x , then M is odd.) Hence we may conclude that

$$\sup_{1 \leq x < y \leq n} |(E(Y(s, n)^m))_{xy}| = O(n^{-3/2}).$$

Let $S(m, n) = S_5(x, x; m, n)$. To complete the proof of Lemma 2.2, it suffices to show that

$$\phi_n(Y(s, n)^m) - S(m, n) = O(n^{-1/2}) \quad (18)$$

as $n \rightarrow \infty$. However (18) follows from (16) once we recall that

$$\phi(Y(s, n)^m) = n^{-1} \sum_{x=1}^n E(Y(s, n)^m)_{xx}.$$

□

Now let us show that each $Y(s, n)$ has a limit distribution.

Lemma 2.3. *For $Y(s, n)$ and in Theorem 2.1, for each $m \geq 1$ there is an integer, α_m , such that for every s ,*

$$\phi_n(Y(s, n)^m) = \alpha_m + O(n^{-1/2})$$

as $n \rightarrow \infty$.

Proof. We saw in Lemma 2.2 that $\phi_n(Y(s, n)^m) = S(m, n) + O(n^{-1/2})$, so it suffices to show

$$S(m, n) = \alpha_m + O(n^{-1/2}), \quad (19)$$

where

$$S(m, n) = n^{-m/2} \sum_{\mathcal{Q} \text{ with } P_5} \theta_{xx}(\mathcal{Q}), \quad (20)$$

and $\theta_{xx}(\mathcal{Q})$ is the cardinality of the set, $W_{\mathcal{Q}}$, of choices of i_2, \dots, i_m which give \mathcal{Q} , (which, as we saw, is independent of $x \in \{1, \dots, n\}$). For fixed \mathcal{Q} (with P_5), let G be the quotient graph obtained in the proof of Lemma 2.2. Either one or two vertices of G are automatically assigned the value x , and there remain $d'(\mathcal{Q})$ vertices of G to which to assign values, (where $d'(\mathcal{Q}) = d(\mathcal{Q}) - 1$ or $d(\mathcal{Q}) - 2$). Then $W_{\mathcal{Q}}$ clearly contains those choices which assign distinct elements of $\{1, \dots, n\}$ to these vertices of G , and is contained in the set of choices which assign arbitrary elements of $\{1, \dots, n\}$ to these vertices of G . Thus we see that $n(n-1) \cdots (n-d'(\mathcal{Q})) \leq \theta_{xx}(\mathcal{Q}) \leq n^{d'(\mathcal{Q})}$. But the difference between these two bounds is $O(n^{d'(\mathcal{Q})-1})$ and $d'(\mathcal{Q}) \leq m/2$ (because G has $m/2$ edges). Hence (19) holds if we set α_m equal to the number of gatherings, \mathcal{Q} with P_5 , for which $d'(\mathcal{Q}) = m/2$. \square

Remark 2.4². One can now show that the distribution $\mu(X^m) = \alpha_m$ is a semicircle law of radius 2. Clearly $\alpha_m = 0$ if m is odd, so let $m = 2h$ be even. Moreover, α_m is equal to the cardinality of the set, \mathcal{C} , of pairings of all the edges (with opposite orientation) of the straight line graph of length m , such that successive clicking suffices to identify all the pairs. For each such pairing, \mathcal{R} , label each edge of the length m straight line graph with $+1$ or -1 as follows: for each $k \sim k'$ with $k < k'$, label the k th edge with -1 and the k' th edge with $+1$. This labeling, which is actually a mapping from $\{1, \dots, m\}$ to $\{\pm 1\}$, we call $\phi(\mathcal{R})$. Let \mathcal{E} be the set of all mappings $L : \{1, \dots, m\} \rightarrow \{\pm 1\}$ such that $\sum_{j=1}^m L(j) = 0$ and $\sum_{j=1}^l L(j) \leq 0 \forall 1 \leq l \leq m$.

ϕ maps \mathcal{C} into \mathcal{E} , and it is actually a bijection. To see this, construct the inverse, ψ , of ϕ , recursively as follows: if L is a labeling, let k be least such that $L(k) = +1$; then necessarily $L(k-1) = -1$; set $k-1 \stackrel{\psi(L)}{\sim} k$, remove the edges $k-1$ and k and repeat the process, searching for the lowest pair of neighboring -1 and $+1$'s, until all the edges have been paired.

Now, setting about determining the size of \mathcal{E} , it is easily seen that the set \mathcal{E} is in bijection with the set of paths of length m from $(0,0)$ to $(m/2, m/2)$ in the lattice $\{0, \dots, m/2\}$ which never go below the diagonal. For example, if $m = 6$, to the labeling $(-1, 1, -1, -1, 1, 1)$ we associate the path shown in Figure 3. The number of such paths can be computed without much difficulty (see for instance [16], problem 81) and is found to be $\binom{2h}{h} - \binom{2h}{h-1}$. Hence it suffices to show that

$$\int_{-2}^2 t^m \sqrt{4-t^2} dt = \begin{cases} 0, & m \text{ odd} \\ 2\pi \left(\binom{2h}{h} - \binom{2h}{h-1} \right), & m = 2h \geq 2, m \text{ even,} \end{cases}$$

which is easily proved by induction.

Lemma 2.5. *Let $Y(s, n)$ and $D(t, n)$ be as in Theorem 2.1. Let $t_1, \dots, t_r \in T$ and suppose that $D(t_\iota, n)$ for $1 \leq \iota \leq r$ is either equal to $I_n \forall n$ or has zero trace $\forall n$. Let $s_1, \dots, s_r \in S$ be such that if $D(t_\iota, n) = I_n$ then $s_{\iota-1} \neq s_\iota$, (where by convention $s_0 = s_r$). Then*

$$\phi_n(D(t_1, n) [Y(s_1, n)^{p_1}]_E \cdots D(t_r, n) [Y(s_r, n)^{p_r}]_E) = O(n^{-1/2}) \quad (21)$$

as $n \rightarrow \infty$.

Proof. Let

$$\begin{aligned} u_\iota &= p_1 + \cdots + p_{\iota-1} + 1 \text{ for } 1 \leq \iota \leq r, \\ m &= p_1 + \cdots + p_r. \end{aligned}$$

Let us use the abbreviation

$$C_\iota \text{ means } a(i_{u_\iota}, j_{u_\iota}; s_\iota, n) a(i_{u_\iota+1}, j_{u_\iota+1}; s_\iota, n) \cdots a(i_{u_\iota+p_\iota-1}, j_{u_\iota+p_\iota-1}; s_\iota, n). \quad (22)$$

²Thanks to Alexandru Nica for showing me this

Each $D(t, n)$ is the sum of its diagonal and off-diagonal parts, hence the left-hand side of equation (21) is equal to a sum of 2^r terms of the form

$$\phi_n(\psi_1(t_1, n) [Y(s_1, n)^{p_1}]_E \cdots \psi_r(t_r, n) [Y(s_r, n)^{p_r}]_E),$$

where each $\psi_\iota(t_\iota, n)$ is either the diagonal or off-diagonal part of $D(t_\iota, n)$, the same for all n . Thus

$$\begin{aligned} & \phi_n(\psi_1(t_1, n) [Y(s_1, n)^{p_1}]_E \cdots \psi_r(t_r, n) [Y(s_r, n)^{p_r}]_E) \\ &= n^{-1} \sum_{\substack{i_1, \dots, i_m \\ j_k = *}} \left(\prod_{1 \leq \iota \leq r} d(j_{u_\iota-1}, i_{u_\iota}; t_\iota, n) \right) E((C_1 - E(C_1)) \cdots (C_r - E(C_r))) \\ &= n^{-1} \sum_{\substack{i_1, \dots, i_m \\ j_k = *}} \left(\prod_{1 \leq \iota \leq r} d(j_{u_\iota-1}, i_{u_\iota}; t_\iota, n) \right) \sum_{f \in \{0,1\}^r} (-1)^{|f|} E(E^{f_1}(C_1) \cdots E^{f_r}(C_r)), \end{aligned} \quad (23)$$

where we are summing over all i_1, \dots, i_m in $\{1, \dots, n\}$, and all allowable j_k , where j_k is allowed to take on only the value i_{k+1} , unless $k = u_\iota - 1$ and $\psi_\iota(t_\iota, n)$ is off-diagonal, in which case we find $0 \leq b \leq (n/N) - 1$ and $1 \leq \hat{i} \leq N$ such that $i_{u_\iota} = Nb + \hat{i}$ and allow j_k to take on all of the values $Nb + \hat{j}$ for $1 \leq \hat{j} \leq N$, $\hat{j} \neq \hat{i}$ (all subscripts of i and j are to be taken modulo m); we are summing over all $f = (f_1, \dots, f_r) \in \{0, 1\}^r$, and define $|f| = f_1 + \cdots + f_r$,

$$E^{f_\iota}(C_\iota) = \begin{cases} C_\iota & \text{if } f_\iota = 0 \\ E(C_\iota) & \text{if } f_\iota = 1. \end{cases}$$

Given a choice of i_1, \dots, i_m and j_1, \dots, j_m , we define its gathering, \mathcal{Q} , as in the proof of Lemma 2.2, (i.e. \mathcal{R} is defined by (10) and \mathcal{O} by (11)). Note that once again, for a choice of i_1, \dots, i_m and j_1, \dots, j_m to give a nonzero term in equation (23), its gathering, \mathcal{Q} , must have property P. Thus we may rearrange the sum to give

$$\begin{aligned} & \phi_n(\psi_1(t_1, n) [Y(s_1, n)^{p_1}]_E \cdots \psi_r(t_r, n) [Y(s_r, n)^{p_r}]_E) \\ &= \sum_{\substack{\mathcal{Q} \text{ with P}}} n^{-1} \sum_{\substack{i_2, \dots, i_m \\ j_k = * \\ \text{giving } \mathcal{Q}}} \left(\prod_{1 \leq \iota \leq r} d(j_{u_\iota-1}, i_{u_\iota}; t_\iota, n) \right) \sum_{f \in \{0,1\}^r} (-1)^{|f|} E(E^{f_1}(C_1) \cdots E^{f_r}(C_r)), \end{aligned} \quad (24)$$

where the sums are over all gatherings, \mathcal{Q} , with P and all choices of i_1, \dots, i_m and all allowable j_1, \dots, j_m in $\{1, \dots, n\}$ giving gathering \mathcal{Q} . Let $\theta(\mathcal{Q})$ be the number of such choices. Using the generalized Hölder inequality together with the boundedness conditions on the moments of the $a(i, j)$'s and on the absolute values of the d 's, we have that for a particular \mathcal{Q} , the corresponding term in equation (24) becomes

$$n^{-1} \sum_{\substack{i_2, \dots, i_m \\ j_k = * \\ \text{giving } \mathcal{Q}}} \left(\prod_{1 \leq \iota \leq r} d(j_{u_\iota-1}, i_{u_\iota}; t_\iota, n) \right) \sum_{f \in \{0,1\}^r} (-1)^{|f|} E(E^{f_1}(C_1) \cdots E^{f_r}(C_r)) = O(n^{-\frac{m}{2}-1})\theta(\mathcal{Q}). \quad (25)$$

We will show that for each \mathcal{Q} with P, the left hand side of (25) is $O(n^{-1/2})$ as $n \rightarrow \infty$.

For each ι with $\psi_\iota(t_\iota, n)$ off-diagonal (and $i_{u_\iota} = Nb + \hat{i}_\iota$), there are $N - 1$ allowable values of $j_{u_\iota-1} = Nb + \hat{j}_\iota$ all nonequal to i_{u_ι} and corresponding to choices of $\hat{i}_\iota - \hat{j}_\iota$. Hence

$$\theta(\mathcal{Q}) = \sum_{\substack{h_\iota \in \{1, \dots, N-1\} \\ \text{for all } \iota \text{ off-diagonal}}} \theta(\mathcal{Q}, h), \quad (26)$$

where each h_ι specifies the value of $\hat{i}_\iota - \hat{j}_\iota$ modulo N , and we denote by h the aggregate of the h_ι 's, for ι running over those values for which ψ_ι is off-diagonal; $\theta(\mathcal{Q}, h)$ is the number of choices of i_1, \dots, i_m such that when j_1, \dots, j_m are then taken according to h , \mathcal{Q} is the resulting gathering; in (26), we are summing over all possible values of $h = (h_\iota)_{\{\iota | \psi_\iota \text{ off-diag.}\}}$. Fixing gathering $\mathcal{Q} = (\mathcal{R}, \mathcal{O})$, let G be the quotient graph obtained from the m -gon graph (see Figure 4) by, as in the proof of Lemma 2.2, identifying edges according to \mathcal{R} with orientations specified by \mathcal{O} , and let $d(\mathcal{Q})$ be the number of vertices of G . So $d(\mathcal{Q})$ is the number of equivalence classes of $\{1, \dots, m\}$ under \sim , defined at (15).

We claim that

$$\theta(\mathcal{Q}, h) \leq n^{d(\mathcal{Q})} \quad (27)$$

for every h . To see this, note that for $k_1 \sim k_2$, i_{k_1} and i_{k_2} differ by a fixed integer (determined by h). Thus there are $d(\mathcal{Q})$ degrees of freedom when choosing values of i_1, \dots, i_m giving gathering \mathcal{Q} , which proves (27).

As an aside, let us remark at this point that for a given choice of \mathcal{Q} and h , the above mentioned fixed differences between i_{k_1} and i_{k_2} for $k_1 \sim k_2$ may be incompatible within an equivalence class of \sim , in which case $\theta(\mathcal{Q}, h) = 0$. This phenomena will be important later.

The case of \mathcal{Q} with property P may be split into the three cases, \mathcal{Q} with P_3 , P_4 and P_5 . If \mathcal{Q} has P_3 , then the quotient graph, G has at most $\frac{m-1}{2}$ edges, so $\frac{m+1}{2}$ vertices. Hence $d(\mathcal{Q}) \leq \frac{m+1}{2}$ and the quantity, (25), is $O(n^{-1/2})$ as $n \rightarrow \infty$.

If \mathcal{Q} has P_4 , then one may click c times to leave a ring with $m - 2c$ vertices and tails with c vertices. (See the proof of Lemma 2.2 for the definition of a click, or just look at Figure 2.) If one has clicked as much as possible and is still left with an inner ring, then each vertex in the inner ring becomes identified with some vertex not equal to itself after the remaining edge identifications are made, so G has at most $\frac{m}{2}$ vertices. If one is able to click so much that the ring disappears, then since by hypothesis one pair of vertices is identified with orientation preserved (and thus can never take part in a click), the only possibility is, before making the last click, to have a square with tails. Now clicking two of the sides of the square and identifying the other two with orientation preserved has the effect of identifying three vertices of the square together. Thus G has at most $\frac{m}{2}$ vertices. Hence the quantity, (25), is $O(n^{-1})$ as $n \rightarrow \infty$.

If \mathcal{Q} has P_5 , then \mathcal{Q} pairs each $a(i, j; s_\iota, n)$ with some $a(j, i; s_{\iota'}, n)$. We may suppose that always $s_{\iota'} = s_\iota$, for otherwise the left hand side of (25) is zero by condition (4). Moreover $E(C_\iota)$ is zero unless \mathcal{Q} pairs $a(i, j; s_\iota, n)$'s only with other $a(i, j; s_\iota, n)$'s from the same C_ι . Such a \mathcal{Q} we say preserves C_ι . Let us say that a given $f \in \{0, 1\}^r$ is compatible with \mathcal{Q} if $f_\iota = 1$ implies that \mathcal{Q} preserves C_ι . Then

$$E(E^{f_1}(C_1) \cdots E^{f_r}(C_r)) = \begin{cases} n^{-m/2} & \text{if } f \text{ is compatible with } \mathcal{Q} \\ 0 & \text{otherwise.} \end{cases}$$

Now it is easy to show that

$$\sum_{f \in \{0, 1\}^r} (-1)^{|f|} E(E^{f_1}(C_1) \cdots E^{f_r}(C_r)) = \begin{cases} n^{-m/2} & \text{if } \mathcal{Q} \text{ preserves no } C_\iota \\ 0 & \text{otherwise.} \end{cases} \quad (28)$$

Thus we are left to show that the left hand side of (25) is $O(n^{-1/2})$ as $n \rightarrow \infty$ for \mathcal{Q} which has P_5 , pairs each random variable $a(i, j; s_\iota, n)$ with its conjugate, and preserves no C_ι . Because \mathcal{Q} doesn't preserve any C_ι , there must be at least one edge in each C_ι which is paired with an edge in some $C_{\iota'}$, for $s_\iota = s_{\iota'}$ and $\iota \neq \iota'$. Suppose first that each $D(t_\iota, n) = I_n$ and hence that $s_{\iota-1} \neq s_\iota$ for every ι . Then between C_ι and $C_{\iota'}$ on the graph, there are always edges belonging to some C_κ , where $s_\iota \neq s_\kappa$. Hence by clicking, we'll never be able to completely remove all the edges of any C_ι from the ring. Thus, as we've seen, clicking as much as possible (c times) will leave $m - 2c$ vertices in the ring and c in the tails, and each of the vertices remaining in the ring will become identified with at least one other vertex not itself. G will thus have at most $m/2$ vertices, and the left hand side of (25) is $O(n^{-1/2})$ as $n \rightarrow \infty$.

Now suppose that some $D(t_\kappa, n)$ has trace zero. We saw in the last paragraph that the quantity (25) is $O(n^{-1})$ unless we suppose that by merely clicking, one may eventually identify an edge of some C_ι with an edge of $C_{\iota-1}$. But because \mathcal{Q} preserves no C_κ , for the first such ι , call it $\tilde{\iota}$, this would imply that the vertex numbered $u_{\tilde{\iota}}$ gets clicked out of the ring into the tails without being identified with any of the vertices, $\{u_\iota \mid \iota \neq \tilde{\iota}\}$. Clearly $s_{\tilde{\iota}} = s_{\tilde{\iota}-1}$ so $D(t_{\tilde{\iota}}, n)$ must have trace zero, and we claim that in addition, $\psi_{\tilde{\iota}}(t_{\tilde{\iota}}, n)$ is the diagonal part of $D(t_{\tilde{\iota}}, n)$. (This is where the aforementioned phenomena of incompatibility of \mathcal{Q} and h comes into the limelight.) Clicking may have occurred within $C_{\tilde{\iota}-1}$ and $C_{\tilde{\iota}}$, and then a click occurs at the boundary between them. Thus, for this boundary-click, an edge of $C_{\tilde{\iota}-1}$, call it $a(i_{k_1}, j_{k_1}; s_{\tilde{\iota}-1}, n)$, is clicked with an edge of $C_{\tilde{\iota}}$, call it $a(i_{k_2}, j_{k_2}; s_{\tilde{\iota}}, n)$; moreover, tracing the history of clicks, we see that $j_{k_1} = j_{u_{\tilde{\iota}-1}}$ and $i_{k_2} = i_{u_{\tilde{\iota}}}$. Then, as $\mathcal{O}(k_1, k_2) = -1$, we have $j_{u_{\tilde{\iota}-1}} = j_{k_1} = i_{k_2} = i_{u_{\tilde{\iota}}}$, which implies that $\psi_{\tilde{\iota}}(t_{\tilde{\iota}}, n)$ is diagonal.

If h is specified and we know the quotient graph G , what do we know about the set of choices of i_1, \dots, i_m which give \mathcal{Q} ? It clearly contains those choices which assign to the vertices of G elements of $\{1, \dots, n\}$ which are pair-wise of distance at least w apart, and is contained in the set of choices which assign arbitrary elements of $\{1, \dots, n\}$ to the vertices of G . Thus we see that $n(n-2w) \cdots (n-d(\mathcal{Q})(2w)) \leq \theta(\mathcal{Q}) \leq n^{d(\mathcal{Q})}$. But the difference between these two bounds is $O(n^{d(\mathcal{Q})-1})$, and as $d(\mathcal{Q}) \leq 1 + m/2$ (because G has $m/2$ edges), it does no harm in equation (25) to sum over those choices of i_1, \dots, i_r which assign arbitrary elements of $\{1, \dots, n\}$ to the vertices of G . However we see from (28) that, in equation (25), since \mathcal{Q} has P_5 and preserves no C_ι , each sum over $f \in \{0, 1\}^r$ has value $n^{-m/2}$. Moreover as we saw in the preceding paragraph, the vertex $u_{\tilde{\iota}}$ is identified with no other u_ι vertex in G , so the quantity $\sum_{1 \leq k \leq n} d(k, k; t_{\tilde{\iota}}, n)$ (which is zero because it equals the trace of $D(t_{\tilde{\iota}}, n)$) factors out of equation (25). Thus even if one can click away the ring, the left hand side of (25) equals zero plus $O(n^{-1})$.

Hence we have proved that each of the terms of equation (24) is $O(n^{-1/2})$. \square

Lemma 2.6. *Let $Y(s, n)$ and $D(t, n)$ be as in Theorem 2.1. Let $s_1, \dots, s_r \in S$, $t_1, \dots, t_r \in T$. Then*

$$\phi_n(D(t_1, n) [Y(s_1, n)^{p_1}]_* \cdots D(t_r, n) [Y(s_r, n)^{p_r}]_*) = O(n^{-1/2}) \quad (29)$$

as $n \rightarrow \infty$, where each $[Y(s_\iota, n)^{p_\iota}]_*$ means either $[Y(s_\iota, n)^{p_\iota}]_E$ or $[Y(s_\iota, n)^{p_\iota}]_r$, the same for all n , and it is $[Y(s_\iota, n)^{p_\iota}]_r$ for at least one ι .

Proof. Let l be the number of $[Y(s_\iota, n)^{p_\iota}]_*$ in equation (29) which are $[Y(s_\iota, n)^{p_\iota}]_r$, (thus $l \geq 1$), and let m be the sum of those p_ι for which $[Y(s_\iota, n)^{p_\iota}]_*$ is $[Y(s_\iota, n)^{p_\iota}]_E$. Write

$$[Y(s_\iota, n)^{p_\iota}]_r = \sum_{1 \leq i, j \leq n} f(i, j; s_\iota, p_\iota, n) e(i, j; n),$$

Then

$$\phi_n(D(1, n) [Y(s_1, n)^{p_1}]_* \cdots D(r, n) [Y(s_r, n)^{p_r}]_*) = n^{-1} \sum_{\substack{i_1, \dots, i_{l+m} \\ j_k = *}} c E(\mathcal{P}), \quad (30)$$

where c (depending on the i 's and j 's) is a product of $d(j_{u_{\iota-1}}, i_{u_\iota}; t_\iota, n)$'s, one for each ι , \mathcal{P} (also depending on the i 's and j 's) is a product of l terms of the form $f(i_k, j_k; s_\iota, p_\iota, n)$ and $r-l$ terms of the form $C_\iota - E(C_\iota)$ and the sum is over all i_1, \dots, i_m and allowable j_1, \dots, j_m in $\{1, \dots, n\}$, where “allowable” is defined as in the proof of Lemma 2.5, modified in the obvious way for this situation. Let L be the set of $k \in \{1, \dots, l+m\}$ such that $f(i_k, j_k; p_\iota, n)$ appears in \mathcal{P} , and let M be the complement of L . Thus L and M have cardinalities l and m , respectively. To find the precise expressions for C_ι as at (22), we would need to redefine u_ι , but the relevant fact here is that C_ι is a product of p_ι random variables of the form $a(i, j; s_\iota, n)$; moreover, the redefinition of the u_ι 's is necessary to correctly describe the rules of “allowable” j_k 's and to define c , but the relevant facts are that for r values of k there are at most N values $j_k - i_{k+1}$ can take, that $j_k = i_{k+1}$ for all other values of $k \in M$, and that c is bounded. As in the proofs of the above lemmas, for each choice of i_1, \dots, i_m and i_1, \dots, i_m , let $\mathcal{Q} = (\mathcal{R}, \mathcal{Q})$ be the associated gathering on M i.e. \mathcal{R} is an

equivalence relation on M indicating how the $a(i, j; n)$'s group together (defined precisely by (10)) and \mathcal{O} indicates the orientations of the identifications, and is defined precisely by (11). As before, for $E(\mathcal{P})$ to be nonzero, \mathcal{Q} must have property P. Moreover, let L_d having cardinality l_d ("d" for diagonal) be the set of $k \in L$ such that $i_k = j_k$ and let L_o having cardinality l_o ("o" for off-diagonal) be the set of $k \in L$ such that $i_k \neq j_k$. Then Lemma 2.2 and the boundedness of moments (3), together with the generalized Hölder inequality imply that

$$E(\mathcal{P}) = O(n^{-(m+l_d+3l_o)/2})$$

as $n \rightarrow \infty$, where $O(n^{-(\dots)/2})$ is independent of i_1, \dots, i_{l+m} and j_1, \dots, j_{l+m} . Thus rearranging the sum in equation (30),

$$\begin{aligned} \phi_n(D(1, n) [Y(s_1, n)^{p_1}]_* \cdots D(r, n) [Y(s_r, n)^{p_r}]_*) &= n^{-1} \sum_{\mathcal{Q} \text{ with P}} \sum_{L_d \subseteq L} \sum_{\substack{i_1, \dots, i_{l+m} \\ j_k = *}} cE(\mathcal{P}) \\ &= n^{-1} \sum_{\mathcal{Q} \text{ with P}} \sum_{L_d \subseteq L} O(n^{-(m+l_d+3l_o)/2}) \theta(\mathcal{Q}, L_d), \end{aligned} \quad (31)$$

where the sums are over all gatherings \mathcal{Q} , on M having property P, all L_d , subsets of L , and all i_1, \dots, i_{l+m} and all allowable j_1, \dots, j_{l+m} in $\{1, \dots, n\}$ which give gathering \mathcal{Q} and diagonal set L_d ; and where $\theta(\mathcal{Q}, L_d)$ is the number of such choices of i_1, \dots, i_{l+m} and j_1, \dots, j_{l+m} .

Now

$$\theta(\mathcal{Q}, L_d) \leq N^r n^{d(\mathcal{Q}, L_d)}, \quad (32)$$

(we take N^r to account for different choices of j_1, \dots, j_{m+l} possible for each i_1, \dots, i_{m+l}), where $d(\mathcal{Q}, L_d)$ is the number of vertices in the quotient graph G of the $(m+l)$ -gon graph, where G is obtained in the following way:

- I) if $k \in L_o$, do nothing to the k th edge,
- II) if $k \in L_d$, remove the k th edge and identify its vertices,
- III) if $k, k' \in M$ and $k \stackrel{\mathcal{R}}{\sim} k'$, then identify the k th and k' th edges with orientation determined by $\mathcal{O}(k, k')$.

Thus G has at most $l_o + (m/2)$ edges, so $d(\mathcal{Q}, L_d) \leq l_o + \frac{m}{2} + 1$. Hence substituting into equations (32) and (31), and noting that either $l_d \geq 1$ or $l_o \geq 1$, one has proved the lemma. \square

Thus the proof of Theorem 2.1 is completed. A special case is

Corollary 2.7. *If N divides n let Ψ_{Nn} embed $M_N(\mathbf{C})$ in $M_n(L)$ by*

$$\Psi_{Nn}(e(\hat{i}, \hat{j}; N)) = \sum_{b=0}^{(n/N)-1} e(Nb + \hat{i}, Nb + \hat{j}; n)$$

for $1 \leq \hat{i}, \hat{j} \leq N$. Let $Y(s, n)$ be random matrices as in Theorem 2.1. Then the family of subsets of random variables,

$$((Y(s, n))_{s \in S}, \Psi_{Nn}(M_N(\mathbf{C})))$$

is asymptotically free as $n \rightarrow \infty$ passing through multiples of N .

As can be seen, in the proofs of Lemmas 2.5 and 2.6, the net effect of N is a constant multiple of N^r in the bounds on the asymptotic behavior of the quantities under consideration. If N were allowed to grow as $n \rightarrow \infty$, the only criteria for the theorem to remain true would be that $N^r O(n^{-1/2})$ go to zero as $n \rightarrow \infty$ (for every r). Hence we may extend Theorem 2.1 to the following:

Theorem 2.8. *Let $Y(s, n)$ be as in Theorem 2.1. Let n_k be a sequence of positive integers tending to ∞ and let N_k dividing n_k for each k be such that $N_k n_k^{-\epsilon}$ goes to zero as $k \rightarrow \infty$ for all $\epsilon > 0$. For each t taking values in some set, T , let*

$$D(t, n_k) = \sum_{\substack{0 \leq b \leq (n_k/N_k)-1 \\ 1 \leq \hat{i}, \hat{j} \leq N_k}} d(N_k b + \hat{i}, N_k b + \hat{j}; t, n_k) e(N_k b + \hat{i}, N_k b + \hat{j}; n_k)$$

be elements of $\Delta(N_k, n_k)$ having a limit distribution as $k \rightarrow \infty$ and such that

$$\sup_{k, i, j} |d(i, j; t, n_k)| < \infty. \quad (33)$$

Assume also that for each $t_1, t_2 \in T$, there exists $t_3 \in T$ such that $D(t_1, n_k)D(t_2, n_k) = D(t_3, n_k) \forall k$. Then the family of sets of random variables

$$((\{Y(s, n_k)\})_{s \in S}, \{D(t, n_k) | t \in T\})$$

is asymptotically free as $k \rightarrow \infty$ and moreover each $Y(s, n_k)$ has limit distribution equal to a semicircle law.

Also in the above, N_k could be allowed to grow more quickly if also the bounds, (33) on $d(i, j; t, n_k)$ went to zero quickly enough as $k \rightarrow \infty$.

§3. Free products of $L(\mathbf{F}_K)$ with a matrix algebra

In this section we use random matrices to investigate $L(\mathbf{F}_K) * M_N(\mathbf{C})$. Henceforth, as pertains to elements of $*$ -noncommutative probability spaces, “semicircular” will mean semicircular of radius 2, and similarly “quarter-circular” and “circular” will mean of radius 2. We shall use the following result, which Voiculescu proved by approximating a semicircular element with random matrices and breaking the $n \times n$ approximants into N^2 ($n/N \times n/N$) blocks.

Proposition 3.1. ([10]). *Let (A, ϕ) be a $*$ -noncommutative probability space and let $D \subseteq A$ be a commutative $*$ -subalgebra. Let A contain*

$$\begin{aligned} \nu &= \{F(p; s) \mid 1 \leq p \leq N, s \in S\} \text{ — a free family of semicircular elements,} \\ \nu' &= \{G(p, q; s) \mid 1 \leq p < q \leq N, s \in S\} \text{ — a } * \text{-free family of circular elements,} \end{aligned}$$

such that $\{\nu, \nu', D\}$ is $$ -free. In $A \otimes M_N(\mathbf{C})$ let*

$$\begin{aligned} X(s) &= N^{-1/2} \left(\sum_{1 \leq p \leq N} F(p; s) \otimes e_{pp} + \sum_{1 \leq p < q \leq N} (G(p, q; s) \otimes e_{pq} + G(p, q; s)^* \otimes e_{qp}) \right) \\ d &= \sum_{1 \leq p \leq N} d(p) \otimes e_{pp}, \end{aligned}$$

where $\{d(p) \mid 1 \leq p \leq N\} \subseteq D$. Then each $X(s)$ is semicircular and $\{d, (X(s))_{s \in S}\}$ is $$ -free in $(A \otimes M_N(\mathbf{C}), \phi \otimes \tau_N)$.*

The next result uses the random matrices with block diagonals of §2.

Proposition 3.2. *Let S be any nonempty set, $N \geq 2$. In a $*$ -noncommutative probability space (A, ϕ) with ϕ a trace, let $\nu_1 = \{X(s) \mid s \in S\}$ be a free family of semicircular elements and $\nu_2 = \{e_{ij} \mid 1 \leq i, j \leq N\}$ a system of matrix units in A such that $\{\nu_1, \nu_2\}$ is free. Let*

$$\begin{aligned} F(k, s) &= N^{1/2} e_{1k} X(s) e_{k1} & (1 \leq k \leq N) \\ G(i, j, s) &= N^{1/2} e_{1i} X(s) e_{j1} & (1 \leq i < j \leq N) \end{aligned}$$

for $s \in S$, $1 \leq i, j, k \leq N$, $i < j$. Then in the $$ -noncommutative probability space, $(e_{11} A e_{11}, N \phi|_{e_{11} A e_{11}})$,*

$$\omega_1 = \{F(k, s) \mid 1 \leq k \leq N : s \in S\}$$

is a free family of semicircular elements and

$$\omega_2 = \{G(i, j, s) \mid 1 \leq i < j \leq N, : s \in S\}$$

is a $*$ -free family of circular elements such that $\{\omega_1, \omega_2\}$ is $*$ -free.

Proof. Model the semicircular family ν_1 and the matrix units as in Corollary 2.7, with the additional stipulation that $\{a(i, j; s, n) \mid 1 \leq i \leq j \leq n, s \in S\}$ be an independent set of random variables in L . Consider the families, $\Omega_1(n)$ and $\Omega_2(n)$ of $(n/N \times n/N)$ matrices which approximate the families ω_1 and ω_2 . Take $\sqrt{2}$ times the real and imaginary parts of the matrices in $\Omega_2(n)$ and apply Theorem 2.1 to conclude that they, together with $\Omega_1(n)$, are asymptotically free, each with semicircular limit distribution. \square

In a similar way we can prove

Proposition 3.3. *Let S be any nonempty set, $N = 2M \geq 4$. In a $*$ -noncommutative probability space (A, ϕ) with ϕ a trace, let $\nu_1 = \{X(s) \mid s \in S\}$ be a free family of semicircular elements and $\nu_2 = \{e_{ij} \mid 1 \leq i, j \leq N\}$ a system of matrix units in A such that $\{\nu_1, \nu_2\}$ is free. Consider the system of 2×2 matrix units $E_{11} = \sum_{k=1}^M e_{kk}$, $E_{12} = \sum_{k=1}^M e_{k, M+k}$, $E_{21} = E_{12}^*$ and $E_{22} = 1 - E_{11}$. Then in the $*$ -noncommutative probability space, $(E_{11}AE_{11}, 2\phi|_{E_{11}AE_{11}})$, the family*

$$(\{E_{11}XE_{11}\}, \{E_{12}XE_{21}\}, \{E_{11}XE_{21}\}, \{e_{pq} \mid 1 \leq p, q \leq M\})$$

is $*$ -free.

Theorem 3.4. $L(\mathbf{F}_K) * M_N(\mathbf{C}) \cong L(\mathbf{F}_{N^2K}) \otimes M_N(\mathbf{C})$.

Proof. By straightforward algebra, (e.g. [10], Lemma 3.1), one can apply Proposition 3.2 to show that

$$(L(\mathbf{F}_K * M_N(\mathbf{C}))_{1/N} \cong L(\mathbf{F}_{N^2K}).$$

\square

Voiculescu proved ([10] Theorem 3.3) that

$$L(\mathbf{F}_{N^2K+1}) \otimes M_N(\mathbf{C}) \cong L(\mathbf{F}_{K+1}), \quad (35)$$

so we might hope to get something similar for $L(\mathbf{F}_{N^2K}) \otimes M_N(\mathbf{C})$ which “approaches” (35) as N gets large. We indeed get such a result using the following proposition, which is little more than the proof of [10] Theorem 3.3 run backwards. The idea is, by conjugating with unitaries, to show that Proposition 3.1 holds also when we add a matrix like one of the $X(s)$ ’s, but with the circular $G(1, q)$ ’s and $G(q, 1)$ ’s for $2 \leq q \leq N$ replaced by quarter-circular elements.

Proposition 3.5. *Let N be a positive integer, S a set (finite or infinite, possibly empty). Let (A, ϕ) be a W^* -probability space with ϕ a trace, such that A contains*

$$\begin{aligned} \nu_1 &= \{a(p) \mid 1 \leq p \leq N\} \text{ — a } * \text{-free family of normal elements,} \\ \nu_2 &= \{f(p) \mid 1 \leq p \leq N\} \text{ — a free family of semicircular elements,} \\ \nu_3 &= \{g(p, q) \mid 2 \leq p < q \leq N\} \text{ — a } * \text{-free family of circular elements,} \\ \nu_4 &= \{b(q) \mid 2 \leq q \leq N\} \text{ — a free family of quarter-circular elements,} \\ \nu_5 &= \{f(p; s) \mid 1 \leq p \leq N, s \in S\} \text{ — a free family of semicircular elements,} \\ \nu_6 &= \{a(n, q; s) \mid 1 \leq n < q \leq N, s \in S\} \text{ — a } * \text{-free family of circular elements} \end{aligned}$$

such that $\{\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6\}$ is $*$ -free. Let $M_N(\mathbf{C})$ have system of matrix units $\{e_{pq} \mid 1 \leq p, q \leq N\}$. In $(A \otimes M_N(\mathbf{C}), \phi \otimes \tau_N)$, let

$$\begin{aligned} H &= \sum_{1 \leq p \leq N} f(p) \otimes e_{pp} + \sum_{2 \leq q \leq N} b(q) \otimes (e_{1q} + e_{q1}) + \sum_{2 \leq p < q \leq N} (g(p, q) \otimes e_{pq} + g(p, q)^* \otimes e_{qp}) \\ C &= \sum_{1 \leq p \leq N} a(p) \otimes e_{pp} \\ X(s) &= \sum_{1 \leq p \leq N} f(p; s) \otimes e_{pp} + \sum_{1 \leq p < q \leq N} (g(p, q; s) \otimes e_{pq} + g(p, q; s)^* \otimes e_{qp}). \end{aligned}$$

Then H and $X(s)$ are semicircular and $\{H, C, (X(s))_{s \in S}\}$ is $*$ -free. Also, the von Neumann algebra generated by C and H contains $1 \otimes M_N(\mathbf{C})$ if the normal elements in ν_1 have disjoint spectra.

Proof. By changing A if necessary, we may assume that there exist a $*$ -free family $\nu = \{U\} \cup \{V_p \mid 2 \leq p \leq N\}$ of Haar unitaries such that $\{\nu, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6\}$ is $*$ -free, and bounded measurable functions h_1, \dots, h_N on the unit circle such that $a(1) = h_1(U)$ and $a(p) = h_p(V_p^* U V_p)$ for $2 \leq p \leq N$. Let

$$W = 1 \otimes e_{11} + \sum_{2 \leq p \leq N} V_p \otimes e_{pp}.$$

We shall show that WHW^* is semicircular and that $\{WHW^*, WCW^*\} \cup \{WX(s)W^* \mid s \in S\}$ is $*$ -free. Note that

$$\begin{aligned} WCW^* &= \sum_{1 \leq p \leq N} h_p(U) \otimes e_{pp} \\ WHW^* &= f(1) \otimes e_{11} + \sum_{2 \leq p \leq N} V_p f(p) V_p^* \otimes e_{pp} + \sum_{2 \leq q \leq N} (b(q) V_q^* \otimes e_{1q} + V_q b(q) \otimes e_{q1}) \\ &\quad + \sum_{2 \leq p < q \leq N} (V_p g(p, q) V_q^* \otimes e_{pq} + V_q g(p, q)^* V_p^* \otimes e_{qp}) \\ WX(s)W^* &= f(1; s) \otimes e_{11} + \sum_{2 \leq p \leq N} V_p f(p; s) V_p^* \otimes e_{pp} \\ &\quad + \sum_{2 \leq q \leq N} (g(1, q; s) V_q^* \otimes e_{1q} + V_q g(1, q; s)^* \otimes e_{q1}) \\ &\quad + \sum_{2 \leq p < q \leq N} (V_p g(p, q; s) V_q^* \otimes e_{pq} + V_q g(p, q; s)^* V_p^* \otimes e_{qp}). \end{aligned}$$

By Proposition 3.1, to prove semicircularity of WHW^* and also $*$ -freeness of $(WHW^*, WCW^*, (WX(s)W^*)_{s \in S})$, it suffices to show that in (A, ϕ) ,

$$\nu_1 = \{f(1)\} \cup \{V_p f(p) V_p^* \mid 2 \leq p \leq N\} \cup \{f(1; s) \mid s \in S\} \cup \{V_p f(p; s) V_p^* \mid 2 \leq p \leq N, s \in S\}$$

is a free family of semicircular elements,

$$\begin{aligned} \nu_2 &= \{b(q) V_q^* \mid 2 \leq q \leq N\} \cup \{V_p g(p, q) V_q^* \mid 2 \leq p < q \leq N\} \cup \\ &\quad \cup \{g(1, q; s) V_q^* \mid 2 \leq q \leq N, s \in S\} \cup \{V_p g(p, q; s) V_q^* \mid 2 \leq p < q \leq N, s \in S\} \end{aligned}$$

is a $*$ -free family of circular elements and that $\{\nu_1, \nu_2, \{U\}\}$ is $*$ -free. But (proceeding as in Voiculescu's proof of Theorem 3.3 in [10]), we have polar decompositions

$$\begin{aligned} V_p g(p, q; s) V_q^* &= (V_p V(p, q; s) V_q^*) (V_q |g(p, q; s)| V_q^*) \\ V_p g(p, q) V_q^* &= (V_p V(p, q) V_q^*) (V_q |g(p, q)| V_q^*) \\ \text{and } g(1, q; s) V_q^* &= (V(1, q; s) V_q^*) (V_q |g(1, q; s)| V_q^*) \end{aligned}$$

(where $g(p, q; s) = V(p, q; s)|g(p, q; s)|$ and $g(p, q) = V(p, q)|g(p, q)|$ are polar decompositions). Now it is a question of showing that

$$\begin{aligned} & \nu_1 \cup \{b(q) \mid 2 \leq q \leq N\} \\ & \cup \{V_q|g(p, q)|V_q^* \mid 2 \leq p < q \leq N\} \cup \\ & \cup \{V_p V(p, q) V_q^* \mid 2 \leq p < q \leq N\} \\ & \cup \{V_q \mid 2 \leq q \leq N\} \cup \\ & \cup \{V_q|g(p, q; s)|V_q^* \mid 1 \leq p < q \leq N, s \in S\} \cup \\ & \cup \{V(1, q; s) V_q^* \mid 2 \leq q \leq N\} \cup \\ & \cup \{V_p V(p, q; s) V_q^* \mid 2 \leq p < q \leq N, s \in S\} \cup \\ & \cup \{U\} \end{aligned}$$

is $*$ -free. This is accomplished by using the functional calculus to replace $f(p)$, $f(p; s)$, $b(q)$, $|g(p, q)|$ and $|g(p, q; s)|$ with equivalent Haar-unitaries (i.e. Haar-unitaries which generate the same von Neumann algebra). Keeping Theorem 1.2 in mind, this reduces the question to one of whether certain elements of a free group form a set of free generators of a subgroup, which has in turn a straightforward solution.

Now supposing that the normal elements in ν_1 have disjoint spectra, we see that $1 \otimes e_{pp}$ are spectral projections of C . But then $b(q) \otimes e_{1q}$ is in the $*$ -algebra generated by C and H , and the unitary part of its polar decomposition is $1 \otimes e_{1q}$. \square

One can now prove results similar to the following.

Corollary 3.6.

$$L(\mathbf{F}_{N^2K}) \otimes M_N(\mathbf{C}) \cong L(\mathbf{F}_K) * L^\infty(\{0\} \cup (1/N, 1)), \quad (36)$$

where $L^\infty(\{0\} \cup (1/N, 1))$ is $L^\infty(\gamma)$ equipped with trace $\int \cdot d\gamma$, and γ is the probability measure having mass $1/N$ on $\{0\}$ and Lebesgue measure on the open interval.

Proof. Let S be a set of cardinality $K - 1$. Then let $L(\mathbf{F}_{N^2K}) \subseteq A$ be generated by a free family of Haar unitaries, $\nu_1 = \{\tilde{u}_2, \dots, \tilde{u}_N\}$ together with the families ν_2, \dots, ν_6 of Proposition 3.5. Let

$$C = 1 \otimes e_{11} + \sum_{2 \leq p \leq N} p \tilde{U}_p \otimes e_{pp}.$$

Then C generates the L^∞ space while H and $\{X(s) \mid s \in S\}$ generate $L(\mathbf{F}_K)$. But since, by the proposition, C and H together generate the matrix units, we see that C , H , and $\{X(s) \mid s \in S\}$ generate all of the left-hand side of (36). \square

Note that in exactly the same way, one obtains

$$L(\mathbf{F}_{N^2K-N+1+j}) \otimes M_N(\mathbf{C}) \cong L(\mathbf{F}_K) * L^\infty(\{0\} \cup \dots \cup \{N-1-j\} \cup (\frac{N-j}{N}, 1)) \quad (37)$$

for $0 \leq j \leq N$, where again each atom has measure $1/N$ and the interval has Lebesgue measure. In [10], Voiculescu did the cases $j = 0$ and $j = N$.

§4. Free products of $L(\mathbf{F}_K)$ with the hyperfinite II_1 -factor

Let R denote the hyperfinite II_1 -factor. Since $L(\mathbf{F}_K) * R$ is an inductive limit of $L(\mathbf{F}_K) * M_{2^k}(\mathbf{C}) \cong L(\mathbf{F}_K) * L^\infty(\{0\} \cup (1/2^k, 1))$ and $L^\infty(\{0\} \cup (1/2^k, 1))$ “approaches” $L(\mathbf{Z})$ as $k \rightarrow \infty$, we could hope that $L(\mathbf{F}_K) * R \cong L(\mathbf{F}_{K+1})$. This is indeed the case, however, as the natural generators for the $L^\infty(\{0\} \cup (1/2^k, 1))$ ’s don’t map nicely to each other under the inductive limit homomorphisms, it seemed intractable to try a proof using inductive limits and thus the following proof, though it is, in the end, an inductive argument, consists of simply exhibiting the requisite free generators

Theorem 4.1. For $1 \leq K \leq \infty$,

$$L(\mathbf{F}_K) * R \cong L(\mathbf{F}_{K+1}),$$

where R is the hyperfinite II_1 -factor.

Proof. Since $*$ is associative, it suffices to show $L(\mathbf{Z}) * R \cong L(\mathbf{F}_2)$. Let X be a semicircular random variable generating $L(\mathbf{Z})$. In the hyperfinite II_1 -factor, R , let Ω_k be the $2^k \times 2^k$ system of matrix units,

$$\Omega_k = \{e(p_1 \cdots p_k, q_1 \cdots q_k; 2^k) \mid p_i, q_i \in \{0, 1\}\},$$

for $k \geq 1$, where $p_1 \cdots p_k$ and $q_1 \cdots q_k$ are binary expansions, such that

$$e(p_1 \cdots p_k, q_1 \cdots q_k; 2^k) = e(p_1 \cdots p_k 0, q_1 \cdots q_k 0; 2^{k+1}) + e(p_1 \cdots p_k 1, q_1 \cdots q_k 1; 2^{k+1})$$

and $\bigcup_{k \geq 1} \Omega_k$ generates R . Let $\mathcal{M} = L(\mathbf{Z}) * R$ with trace ϕ .

Let \mathcal{A}_k denote the algebra $e(0 \cdots 0, 0 \cdots 0; 2^k) \mathcal{M} e(0 \cdots 0, 0 \cdots 0; 2^k)$, containing

$$F(p_1 \cdots p_k; k) = 2^{k/2} e(0 \cdots 0, p_1 \cdots p_k; 2^k) X e(p_1 \cdots p_k, 0 \cdots 0; 2^k)$$

$$G(p_1 \cdots p_k, q_1 \cdots q_k; k) = 2^{k/2} e(0 \cdots 0, p_1 \cdots p_k; 2^k) X e(q_1 \cdots q_k, 0 \cdots 0; 2^k).$$

Then by Proposition 3.2,

$$\nu_1 = \{F(p_1 \cdots p_k; k) \mid p_i \in \{0, 1\}\}$$

is a free semicircular family and

$$\nu_2 = \{G(p_1 \cdots p_k, q_1 \cdots q_k; k) \mid p_i, q_i \in \{0, 1\}, p_1 \cdots p_k > q_1 \cdots q_k\}$$

a $*$ -free circular family in $(\mathcal{A}_k, 2^k \phi|_{\mathcal{A}_k})$ such that $\{\nu_1, \nu_2\}$ is $*$ -free. If $a \in \mathcal{A}_k$ let $a \otimes e(p_1 \cdots p_k, q_1 \cdots q_k; 2^k)$ denote $e(p_1 \cdots p_k, 0 \cdots 0; 2^k) a e(0 \cdots 0, q_1 \cdots q_k; 2^k) \in \mathcal{M}$.

Let $G(0 \cdots 01, 0 \cdots 0; k) = V(k)B(k)$ be the polar decomposition, and let

$$\begin{aligned} H &= \sum_{k=1}^{\infty} 2^{-k/2} (F(0 \cdots 01; k) \otimes e(0 \cdots 01, 0 \cdots 01; 2^k) \\ &\quad + B(k) \otimes (e(0 \cdots 0, 0 \cdots 01; 2^k) + e(0 \cdots 01, 0 \cdots 0; 2^k))) \end{aligned}$$

$$W = \sum_{k=1}^{\infty} \frac{1}{k} V(k) \otimes e(0 \cdots 01, 0 \cdots 01; 2^k),$$

(see Figure 5).

It is not hard to show that H and W together generate $L(\mathbf{Z}) * R$. Indeed, from spectral projections of W , one has many of the diagonal matrix units, enough to extract from H the operators, $B(k) \otimes e(0 \cdots 0, 0 \cdots 01; 2^k)$ for $1 \leq k < \infty$. Taking the polar decompositions of these, we get as polar parts $1 \otimes e(0 \cdots 0, 0 \cdots 01; 2^k)$. The ensemble of these, together with the aforementioned spectral projections of W , generate all the matrix units, hence R . Now that we have the matrix units, we can pull apart W and H , and piece the parts together to obtain X (as a norm limit).

Now we shall show that H is semicircular and $\{W, H\}$ is $*$ -free. This will prove the theorem because the spectral measure of W has no atoms, so each of W and H will generate a copy of $L(\mathbf{Z})$. For $n \geq 1$, let

$$\begin{aligned} H_n &= \left(\sum_{k=1}^n 2^{-k/2} (F(0 \cdots 01; k) \otimes e(0 \cdots 01, 0 \cdots 01; 2^k) \right. \\ &\quad \left. + B(k) \otimes (e(0 \cdots 0, 0 \cdots 01; 2^k) + e(0 \cdots 01, 0 \cdots 0; 2^k))) \right) \\ &\quad + 2^{-n/2} F(0 \cdots 0; n) \otimes e(0 \cdots 0, 0 \cdots 0; 2^n) \\ W_n &= \left(\sum_{k=1}^n \frac{1}{k} V(k) \otimes e(0 \cdots 01, 0 \cdots 01; 2^k) \right) + \left(\frac{1}{n+1} \right) 1 \otimes e(0 \cdots 0, 0 \cdots 0; 2^n). \end{aligned}$$

Since $F(\cdots; k)$ has the same norm for all k , and similarly for $G(\cdots; k)$, clearly W_n and H_n converge in norm to W and H . Hence it suffices to show that for each n , H_n is semicircular and $\{W_n, H_n\}$ is $*$ -free. This we do by induction on n . In the case $n = 1$ we have

$$H_1 = 2^{-1/2} \begin{pmatrix} F(0; 1) & B(1) \\ B(1) & F(1; 1) \end{pmatrix}, \quad W_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & V(1) \end{pmatrix}.$$

Then by Propositions 3.5 and 3.2 and Theorem 1.2, H_1 is semicircular and $\{W_1, H_1\}$ is $*$ -free.

For the inductive step, suppose the proposition is true for $n - 1$. Let

$$K_n = 2^{1/2}(H_n - F(1; 1) \otimes e(1, 1; 2) - B(1) \otimes (e(1, 0; 2) + e(0, 1; 2)))$$

$$Y_n = W_n - V(1) \otimes e(1, 1; 2).$$

Then by inductive hypothesis applied to \mathcal{A}_1 , we have that in $(\mathcal{A}_1, 2\phi|_{\mathcal{A}_1})$, K_n is semicircular and $\{K_n, Y_n\}$ is $*$ -free. Note that

$$H_n = 2^{-1/2} \begin{pmatrix} K_n & B(1) \\ B(1) & F(1; 1) \end{pmatrix}, \quad W_n = \begin{pmatrix} Y_n & 0 \\ 0 & V(1) \end{pmatrix}.$$

In $(\mathcal{A}_1, 2\phi)$, let $S = \{e(0p_1 \cdots p_{k-1}, 0q_1 \cdots q_{k-1}; 2^k) \mid p_i, q_i \in \{0, 1\}, 2 \leq k \leq n\}$. Then by Proposition 3.3, $\{S, F(0; 1), G(1, 0; 1), F(1; 1)\}$ is $*$ -free. But K_n and Y_n are generated by S and $F(0; 1)$, so using Proposition 1.1 and Theorem 1.2, we obtain that $\{K_n, Y_n, V(1), B(1), F(1; 1)\}$ is $*$ -free. Thus by Proposition 3.5, H_n is semicircular and $\{W_n, H_n\}$ is $*$ -free. \square

Corollary 4.2. *For $1 \leq n < \infty$, let $(G_i)_{i=1}^n$ be a family of discrete groups such that for each i , G_i is either an amenable i.c.c. group or a copy of \mathbf{Z} , and at least one of them is the latter. Then*

$$L\left(\bigstar_{i=1}^n G_i\right) \cong L(\mathbf{F}_n). \quad (38)$$

Moreover, suppose now that $(G_i)_{i=1}^\infty$ is a family of amenable i.c.c. discrete groups. Then

$$L(\mathbf{F}_\infty * \bigstar_{i=1}^\infty G_i) \cong L(\mathbf{F}_\infty). \quad (39)$$

Proof. Let us first recall Connes' result [3] that whenever G is an amenable i.c.c. discrete group, $L(G)$ is a copy of R , the hyperfinite II_1 -factor. Now (38) follows immediately. For (39), note that free products may be taken in any order, so

$$L(\mathbf{F}_\infty * \bigstar_{i=1}^\infty G_i) \cong L\left(\bigstar_{i=1}^\infty (\mathbf{Z} * G_i)\right) \cong \bigstar_{i=1}^\infty L(\mathbf{Z} * G_i) \cong \bigstar_{i=1}^\infty L(\mathbf{F}_2) \cong L(\mathbf{F}_\infty).$$

\square

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Appendix

As F. Rădulescu observed in [17], there are proofs of Gaussian cases of Theorem 2.1 which are very similar to Voiculescu’s proofs for his random matrix results in [11]. Consider for example the two cases having the following additional hypotheses for Theorem 2.1.

- (I) Real Gaussian: each $a(i, j; s, n)$ is a real Gaussian random variable and $\{a(i, j; s, n) \mid s \in S, 1 \leq i \leq j \leq n\}$ is an independent family.
- (II) Complex Gaussian: each $a(k, k; s, n)$ is a real Gaussian random variable, each $\operatorname{Re} a(i, j; s, n)$ and $\operatorname{Im} a(i, j; s, n)$ for $i \neq j$ are real Gaussian random variables having second moments equal to $1/2n$ and

$$\{\operatorname{Re} a(i, j; s, n) \mid s \in S, 1 \leq i \leq j \leq n\} \cup \{\operatorname{Im} a(i, j; s, n) \mid s \in S, 1 \leq i < j \leq n\}$$

is an independent family.

We remark that the complex Gaussian case alone suffices to prove Proposition 3.2.

In order to prove 2.1 for these two cases, one proceeds exactly as in the proof of Theorem 2.2 of [11], which utilizes Theorem 2.1 of [11] and the fact that a sum of Gaussian random variables is Gaussian to reduce the proof to checking certain second order freeness conditions. These are in turn checked in our case of having block diagonal constant matrices by using counting arguments only slightly more complicated than those in [11].

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